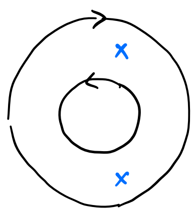
TALK 1: Arnold Conjecture and Morse Homology ①

* Poincare - Birkhoff theorem (1912, 1913)

Every area-preserving, orientation-preserving homeomorphism of an annulus that rotates the two boundaries in opposite directions has at least two fixed points.



* Generalizations to higher dimensions

- Symplectomorphisms are a generalization of area preserving diffeos.

  $\varphi : (M_1, \omega_1) \longrightarrow (M_2, \omega_2)$ diffeomorphism

  s.t. $\varphi^* \omega_2 = \omega_1$.

- Symplectomorphisms need not have fixed points:

  eg: $\mathbb{T}^2 \longrightarrow \mathbb{T}^2$ rotation.

  $(e^{i\theta_1}, e^{i\theta_2}) \longmapsto (e^{i\theta_1}, e^{i(\theta_2 + \pi)})$

- However, Hamiltonian symplectomorphisms on $\mathbb{T}^2$ always do. These correspond to the critical points of the generating Hamiltonian.

Arnold Conjecture (1960s)

A symplectomorphism that is generated by a time-dependent Hamiltonian vector field should have at least as many fixed points as a function on the manifold must have critical points.

Progress :

Still open in full generality as minimal no. of critical points is hard to estimate.

Conjecture (Arnold) Let (M, ω) be a compact symplectic manifold and $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent 1-periodic Hamiltonian function. Suppose that the 1-periodic orbits of H are all nondegenerate. Then

$$\# P(H) \geq \sum_{i=0}^{2n} \dim H_i(M, \mathbb{Q})$$

where $H_i(M, \mathbb{Q})$ denotes the singular homology of M with rational coefficients.

- $H_t : M \to \mathbb{R} \quad \rightsquigarrow \quad X_{H_t} : M \to TM \quad$ s.t. $\iota(X_H)\omega = dH$

  1-parameter family of symplectomorphisms

  $\varphi_t : M \to M$ satisfying $\quad \dfrac{d\varphi_t}{dt} = X_t \circ \varphi_t \qquad \varphi_0 = id$

- The fixed points of the time 1 map φ_1 are in 1-1 corresp. with 1-periodic orbits ie. 1-periodic solutions of

  $\dot{x}(t) = X_t(x(t))$.

  A periodic solution x is nondegenerate if
  $\det(\mathbb{1} - d\varphi_1(x(0))) \neq 0$.

- Arnold conjecture in the above form is proven in all generality:

Eliashberg (1979) : for Riemann surfaces ie. 2 dim.
(Strictly 2-dim techniques)

Conley – Zehnder (1983) : for Torii (of all dimensions) $\mathbb{T}^{2n}$
↳ was extended by many people.
(finite dim. approximation of the symplectic action functional on the loop space.)

Floer (1989) : for Monotone Symplectic manifolds.
(Infinite dimensional Morse theory = Floer theory)

↳ Fukaya – Ono (1996)
Liu – Tian (1996)
Hofer – Salamon (1997–98)

Establish nondegenerate case in homological form for all symplectic manifolds.

⊛ Morse – Smale – Witten Complex

- M a smooth compact Riemannian manifold.
- $f : M \to \mathbb{R}$ a Morse function :

  - A critical point is $x \in M$ such that $(df)_x = 0$.
  - $\mathrm{Crit}(f) = \{ x \in M \mid d_x f = 0 \}$
  - f is said to be Morse if the $\underset{\overset{\shortparallel}{d^2 f_x}}{\mathrm{Hess}_x(f)}$ is nondegenerate (as a bilinear form) at every $x \in \mathrm{Crit}\, f$.

- <u>Thm</u> : For a compact mfld M, the set of Morse fns on M is a open dense subset $C^\infty(M;\mathbb{R})$.

- <u>Morse lemma</u> : Let $c \in \mathrm{Crit}(f)$ for f Morse.
  $\exists$ chart $(U, x_1, \dots, x_n) \subset M$
  with $c = (0, \dots, 0)$. s.t.
  $$f(x_1, \dots, x_n) = f(c) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2.$$

  k depends only on c.
  $k = \mathrm{ind}(c)$. Denote $\mathrm{Crit}_k(f) = \{ c \in \mathrm{Crit}(f) \mid \mathrm{ind}(c) = k \}$.

- Let g be a Riemannian metric on M.
  Then we get a gradient ∇f by $\iota(\nabla f) g = df$.

- Consider the negative gradient flow
  $\varphi : \mathbb{R} \times M \to M$ $\qquad$ $\varphi^s : M \to M$ is for fixed $s \in \mathbb{R}$.
  $$\frac{d\varphi}{dt} = -\nabla f \circ \varphi$$

- Stable manifold $W^s(x) = \{y \in M : \lim_{s\to\infty} \varphi^s(y) = x\}$

  Unstable manifold $W^u(x) = \{y \in M : \lim_{s\to -\infty} \varphi^s(y) = x\}$

  ↳ These are smooth manifolds (Morse lemma + flows) of

  $$\dim W^u(x) = \operatorname{ind}(x)$$

  $$\dim W^s(x) = n - \operatorname{ind}(x).$$

- The pair (f,g) is called Morse-Smale if all the stable and unstable intersect transversely.

- If (f,g) is Morse-Smale then

  $$M(x,y) = W^u(x) \cap W^s(y)$$

  $$= \{z \in M : \lim_{s\to -\infty} \varphi^s(z) = x, \lim_{s\to +\infty} \varphi^s(z) = y\}$$

  is a smooth manifold of dimension

  $$\dim M(x,y) = \operatorname{ind}(x) - \operatorname{ind}(y).$$

- We can look at $\mathcal{M}(x,y) = \left\{ \gamma : \mathbb{R} \to M \mid \begin{array}{l} \dot\gamma = -\nabla f \circ \gamma \\ \lim_{s\to -\infty} \gamma(s) = x \\ \lim_{s\to +\infty} \gamma(s) = y \end{array} \right\}$

  by identifying a flow line γ with $\gamma(0) \in M$.

- Note that $\mathbb{R}$ acts on $\mathcal{M}(x,y)$ by translation.

$$\dot{\gamma}(s) = -\nabla f(\gamma(s)) \Rightarrow \dot{\gamma}(s+c) = -\nabla f(\gamma(s+c))$$

$$\Rightarrow \dot{\gamma}_{+c}(s) = -\nabla f(\gamma_{+c}(s)) \Rightarrow \gamma_{+c} \in \mathcal{M}(x,y).$$

Turns out this is a free action if $x \neq y$.

- Let $\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$. This is the space of unparametrized flow lines from x to y.

$\dim \hat{\mathcal{M}}(x,y) = \operatorname{ind}(x) - \operatorname{ind}(y) - 1.$

- Chain complex :

$$CM_k(f) = \bigoplus_{x \in \operatorname{Crit}_k(f)} \mathbb{Z}/2\mathbb{Z}\langle x\rangle$$

- Differential :

$$\partial_g(x) = \sum_{\operatorname{ind}(y) = \operatorname{ind}(x)-1} \# \hat{\mathcal{M}}(x,y)\langle y\rangle \qquad \text{extend linearly.}$$

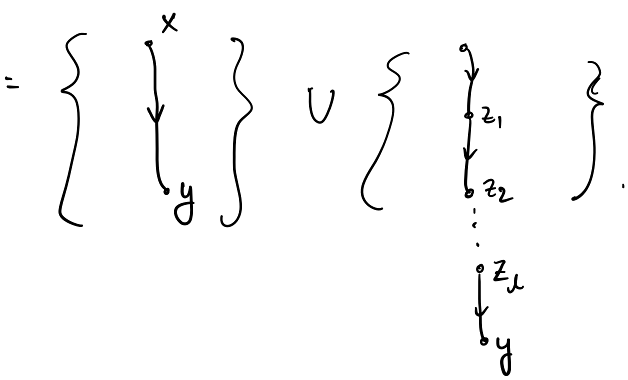
- This is actually a differential ie $\partial_g^2 = 0$.

$$\partial^2\langle x\rangle = \partial\left(\sum_{\operatorname{ind}(y)=\operatorname{ind}(x)-1} \# \hat{\mathcal{M}}(x,y)\cdot\langle y\rangle\right)$$

$$= \sum_{\substack{\operatorname{ind}(y) \\ =\operatorname{ind}(x)-1}} \# \hat{\mathcal{M}}(x,y)\left(\sum_{\substack{\operatorname{ind}(z) \\ =\operatorname{ind}(y)-1}} \cdot \# \hat{\mathcal{M}}(y,z) \cdot \langle z\rangle.\right)$$

$$= \sum_{\substack{\mathrm{ind}(z) \\ = \mathrm{ind}(x)-2}} \left(\sum_{\substack{\mathrm{ind}(y) \\ = \mathrm{ind}(x)-1}} \# \hat{\mathcal{M}}(x,y) \; \# \hat{\mathcal{M}}(y,z) \right) \cdot \langle z \rangle .$$

- Define $\bar{\mathcal{M}}(x,y) = \hat{\mathcal{M}}(x,y) \cup \left(\hat{\mathcal{M}}(x,z_1) \times \hat{\mathcal{M}}(z_1,z_2) \times \cdots \times \hat{\mathcal{M}}(z_l, y) \right)$
$z_i \in \mathrm{Crit}(f)$.



Note that if $\mathrm{ind}(x) \leq \mathrm{ind}(y)$, then $\hat{\mathcal{M}}(x,y) = \phi$.

- Endow $\bar{\mathcal{M}}(x,y)$ with the topology coming from C^∞-convergence on compact subsets of $\mathbb{R}$ upto translation.

$\gamma_\nu \longrightarrow \gamma = (\gamma^1, \ldots, \gamma^l)$ broken trajectory if

for each $j = 1, \ldots, l$, $\tau_\nu^j \in \mathbb{R}$ such that

$\tau_\nu^j \bullet \gamma_\nu \longrightarrow \gamma^j$ in C^∞_{loc}-topology of $C^\infty(\mathbb{R}, M)$.

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- Theorem: $\bar{\mathcal{M}}(x,y)$ is $\underbrace{\text{compact. manifold}}_{\to \text{ compactness}}$ with boundary

  with $\dim \bar{\mathcal{M}}(x,y) = \operatorname{ind}(x) - \operatorname{ind}(y) - 1$ $\rightarrow$ index calculation.

  $$\underbrace{\partial \bar{\mathcal{M}}(x,y) = \bar{\mathcal{M}}(x,y) \setminus \hat{\mathcal{M}}(x,y).}_{\to \text{ gluing.}}$$

- If $\operatorname{ind}(z) = \operatorname{ind}(x) - 2$, $\bar{\mathcal{M}}(x,z)$ is a 1-dim mfld

  then $$\partial \bar{\mathcal{M}}(x,z) = \bigcup_{\operatorname{ind}(y) = \operatorname{ind}(x)-1} \mathcal{M}(x,y) \times \mathcal{M}(y,z)$$

$$\therefore \quad \partial^2(x) = \sum_{\operatorname{ind}(z) = \operatorname{ind}(x)-2} \# \partial \bar{\mathcal{M}}(x,z) \cdot \langle z \rangle = 0.$$

* Good Facts:

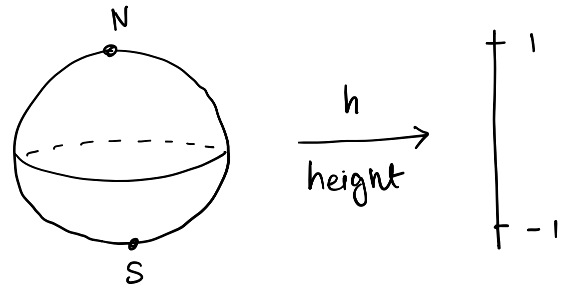
- $HM_*(f, g; \mathbb{Z}/2)$ does not depend on choice of f and g.
- In fact $HM_*(f, g; \mathbb{Z}/2) \cong H_*(M; \mathbb{Z}/2)$ singular homology.
- Morse inequalities

  $$\# \operatorname{Crit}_k(f) \geq \beta_k = \dim H_k(M; \mathbb{Z}/2).$$

- Can define orientations and make $\mathbb{Z}$ coordinates.

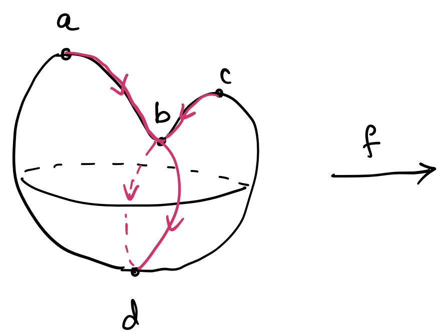
* Examples :

1.



$$C_0(h) = \mathbb{Z}/_2\langle S\rangle \qquad H_0(S^2) \cong \mathbb{Z}/2$$
$$C_1(h) = 0 \qquad H_1(S^2) \cong 0$$
$$C_2(h) = \mathbb{Z}/_2\langle N\rangle \qquad H_2(S^2) \cong \mathbb{Z}/_2$$

2.



$$C_0(h) = \mathbb{Z}/_2\langle d\rangle \qquad \partial d = 0 \qquad H_0 \cong \mathbb{Z}/2$$
$$C_1(h) = \mathbb{Z}/_2\langle b\rangle \qquad \partial b = 2d = 0 \qquad H_1 \cong 0$$
$$C_2(h) = \mathbb{Z}/_2\langle a, c\rangle \qquad \partial a = \partial c = b \qquad H_2 \cong \mathbb{Z}/_2$$

3.
$$H_k(S^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, n \\ 0 & \text{ow} \end{cases}$$

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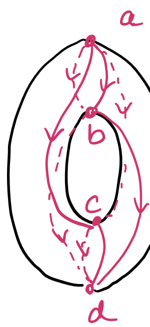
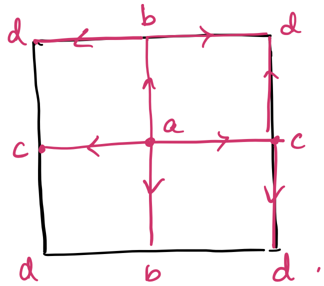




$\partial = 0$ on all chains.

$H_0(\mathbb{T}^2) = \mathbb{Z}/2\langle d \rangle \cong \mathbb{Z}/2$

$H_1(\mathbb{T}^2) = \mathbb{Z}_2\langle c, b \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$H_2(\mathbb{T}^2) = \mathbb{Z}_2\langle a \rangle \cong \mathbb{Z}_2$

TALK 2 Hamiltonian Floer Homology

(M, ω) compact symplectic manifold.

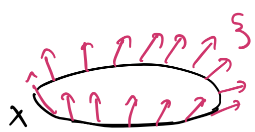
$\mathcal{L}M$ = space of contractible loops on M

$$= \left\{ x : \mathbb{R} \to M \ : \ \begin{array}{l} x(t+1) = x(t) \quad t \in \mathbb{R} \\ x \text{ contractible} \end{array} \right\}$$

$$= \{ x : \underset{\underset{S^1}{\parallel}}{\mathbb{R}/\mathbb{Z}} \to M \ : \ x \text{ contractible} \} + C^\infty \text{ topology}$$

$$T_x \mathcal{L}M = C^\infty(S^1, x^*TM)$$

$$= \{ \xi : S^1 \to TM \mid \xi(t) \in T_{x(t)}M \}.$$



Suppose $H_t = H_{t+1} : M \to \mathbb{R}$ is a 1-periodic time dependent Hamiltonian.

Assumption 1. $\forall$ C^∞ map $u : S^2 \to M$, $\int_{S^2} u^* \omega = 0$

$$\langle \omega, \pi_2(M) \rangle = 0.$$

Assumption 2: $\forall$ C^∞-map $u : S^2 \to M$, $\exists$ a symplectic trivialization of the fiber bundle

$f : u^*TM \to S^2 \times F$
isomorphism
F symplectic sp.

$$\begin{array}{c} u^*TM \\ \downarrow \\ S^2 \end{array} \quad \text{ie} \quad \langle c_1(TM), \pi_2(M) \rangle = 0.$$

$c_i(u^*TM) \in H^{2i}(S^2) = 0$ for $i > 1$.

* Monotonicity condition

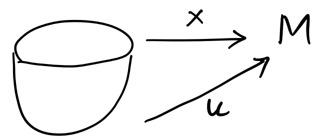
Assume there exists some constant $\tau \in \mathbb{R}$ s.t. for every smooth map $v: S^2 \to M$

$$\int_{S^2} v^* c_1 = \tau \int_{S^2} v^* \omega .$$

$c_1 = c_1(TM, J) \in H^2(M, \mathbb{Z})$ is independent of the choice of $J \in \mathcal{J}(\omega)$.

The (M, ω) is said to be monotone if $\tau > 0$.

* Action functional :



$$\mathcal{A}_H : \mathcal{L}M \longrightarrow \mathbb{R}$$

$$\mathcal{A}_H(x) = -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\,dt$$

u is an extension of x to $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$

$u(e^{2\pi i t}) = x(t)$.

Assumption 2 $\Rightarrow$ $\mathcal{A}_H(x)$ is independent of choice of u.

<u>Prop</u>: A loop x is a critical point of $\mathcal{A}_H$ iff $t \mapsto x(t)$ is a periodic solution of Hamiltonian system

$$\dot{x}(t) = X_t(x(t)).$$

Need computation

$$(d\mathcal{A}_H)_x(Y) = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x), Y)\,dt \quad \forall Y \in T_x\mathcal{L}M$$

$$(d\mathcal{A}_H)_x \equiv 0 \iff \omega(\dot{x}(t) - X_{H_t}(x), Y) = 0 \quad \forall Y$$

$$\iff \dot{x}(t) - X_{H_t}(x) = 0.$$

<u>Defn</u>: A periodic solution x is called nondegenerate if

$$\det(\mathrm{Id} - T_{x(0)}\psi^1) \neq 0.$$

where $\psi^1 : M \to M$ is the time 1 flow of H.

- We will call a critical point x of $\mathcal{A}_H$ nondegenerate if it is a nondeg. periodic solution.

- We may assume (upto C^2-small perturbation of H) that if x and y are geometrically distinct, $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$.

* The Gradient

Fix an almost complex structure J on M compatible with ω, ie.

$\Rightarrow$ $g(v,w) := \omega(v, Jw)$ defines a Riemannian metric.

$\Rightarrow$ Define metric of $\mathcal{L}M$

$$\langle Y, Z\rangle = \int_0^1 g(Y(t), Z(t))\, dt \quad \rightsquigarrow L^1 \text{ metric}$$

Wrt this metric if $f : \mathcal{L}M \to \mathbb{R}$ is a function, we can define the gradient by

$$\begin{aligned}\langle \operatorname{grad}_x f, Y\rangle &= \int_0^1 g_{x(t)}\big((\operatorname{grad}_x f)(t), Y(t)\big)\, dt \\ &= \int_0^1 \omega_{x(t)}\big((\operatorname{grad}_x f)(t), JY(t)\big)\, dt \\ &= (df)_x Y\end{aligned}$$

- $$(d\mathcal{A}_H)_x Y = \int_0^1 \omega_{x(t)}\big(\underbrace{\dot{x}(t) - X_{H_t}(x)}_{-J \operatorname{grad}}, Y(t)\big)\, dt$$

$$\Rightarrow \quad \mathcal{X}_H(t) = -(\operatorname{grad}_x \mathcal{A}_H)(t) = -J(\dot{x}(t)) + J X_{H_t}(x)$$
$$= -J_{x(t)}\, \dot{x}(t) - \nabla_{x(t)} H_t$$

- The negative gradient trajectories or flow lines for $\mathcal{X}_H$ are solutions to $\mathbb{R} \to \mathcal{L}M$, $s \mapsto u(s)$

satisfying $u(s) : S^1 \to M \quad \forall s \in \mathbb{R}$

$$\frac{\partial u}{\partial s}(t) = \mathcal{X}_H(t) = -J_{u(s,t)}\left(\frac{\partial u}{\partial t}\right) - \nabla_{u(s,t)} H_t(u(s,t)).$$

$$\boxed{\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla_{u(s,t)} H_t(u) = 0.}$$ Floer equation.

* Energy

Define
$$E(u) = \int_{-\infty}^{+\infty} \frac{d}{ds} A_H(u(s))\, ds$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{S^1} \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_t(u)\right|^2 \right) ds\, dt$$
$$= \int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt$$
for u solution of Floer equation.

<u>Properties</u>

(1) $E(u) \geq 0$

(2) $E(u) = 0 \iff \frac{\partial u}{\partial s} = 0$ and u solution of Floer eq. $\iff$ u constant path at critical pt of A_H.

(3) If u solution s.t. $\lim_{s\to-\infty} u_s(t) = x(t)$, $\lim_{s\to\infty} u_s(t) = y(t)$,

the $E(u) = A_H(x) - A_H(y) < \infty$

* Moduli space

$$\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+ ; H, J)$$

$$= \left\{ u : \mathbb{R}\times S^1 \longrightarrow M \;\middle|\; \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0 \quad \lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t) \right\}$$

Note: If $H = H_t$ ie time independent, then

and u grad. traj $\frac{\partial u}{\partial s} - \nabla H(u) = 0$.

Let $u(s,t) = u(s)$ ie. $t \in S^1$ independent.

$$\Longrightarrow \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - \nabla H(u) = 0$$

Thm: There exists a subset $H_{reg}(J) \subset C^\infty(M\times S^1, \mathbb{R})$ of second category (countable intersection of open and dense sets) such that all periodic orbits are nondegenerate and

$\mathcal{M}(x^-, x^+ ; H, J)$ is a finite dimensional smooth mfld $\forall\, x^{\pm} \in P(H)$, $\forall\, H \in \mathcal{H}_{reg}$.

Moreover, if (M,ω) monotone $\exists$ function $\eta_H : P(H) \to \mathbb{R}$ s.t. $\forall\, u \in \mathcal{M}(x^-, x^+; H, J)$ the dimension of the moduli space locally near u

$$\dim_u \mathcal{M}(x^-, x^+ ; H, J) = \mu(u; H) = \eta_H(x^-) - \eta_H(x^+) + 2\tau E(u).$$

locally near u.

* Conley - Zehnder index.

- Denote $Sp(2n)$ = group of symplectic $2n\times 2n$ - matrices

  $Sp^*(2n)$ = open and dense set of symp. $2n\times 2n$ matrices which do not have 1 as an eigenvalue

  ie. $\det(\mathbb{1}-\Psi)\neq 0$

  Maslov cycle $= Sp(2n)\setminus Sp^*(2n)$ ie $\det(\mathbb{1}-\Psi)=0$.

  ↳ algebraic vareity of codim 1.

  Suppose $\phi: S^1 \to Sp(2n)$ is a loop, then

  $\Phi\cap$ Maslov cycle is always even.

  Let $\mu(\Phi) = \frac{1}{2}(\#\, \varphi \cap \text{Maslov cycle})$.

- $\mu(\Phi) = \deg(\rho\circ\Phi)$

  where $\rho: Sp(2n)\to S^1$ is continuous extension of

  $\det: U(n) = Sp(2n)\cap O(2n) \to S^1$.

- $Sp(2n)^*$ has two connected component.

  Let $W^+ = -\mathbb{1} \in \{\det > 0\} \subset Sp(2n)^*$

  $W^- = \mathrm{diag}(2,-1,\dots,-1,1/2,-1,\dots,-1) \in \{\det < 0\}$.

  $\rho(W^\pm) = \pm 1$. For an orbit, the differential of the flow defines such a path is $Sp(2n)$.

  For any path $\Psi: [0,1] \to Sp(2n)$, s.t. $\Psi(0) = W^+$, complete Ψ by adding a path to $W^\pm$ according to whether $\det(\Psi(1)) > 0$ or < 0. Then $\rho^2\circ\Psi$ is a loop.

Conley-Zehnder index : $\mu_{CZ}(\Psi) = \deg(\rho^2\circ\Psi)$.

TALK 3 : Hamiltonian Floer continued & Arnold conjecture

* Compactness

Denote by $M^1(x^-, x^+; H, J) = \{u \in M(x^-, x^+; H, J) : \mu(u; H) = 1\}$

↳ it is the one-dimensional part of $M(x^-, x^+)$.

Prop: If (M,ω) monotone and $H \in H_{reg}$ then

$$\hat{M}^1(x^-, x^+; H, J) = M^1(x^-, x^+; H, J)/\mathbb{R}$$

is a finite set $\forall\, x^\pm \in P(H)$.

Lemma: Suppose (M,ω) is monotone with minimal Chern number N, then for $\hbar = N/2$, $\wedge$ nonconstant $v : S^2 \longrightarrow M$ satisfying $\bar{\partial}_J(v) = \frac{1}{2}(dv + J \circ dv \circ i) = 0$

$$E(v) = \int_{S^2} v^*\omega \geq \hbar .$$

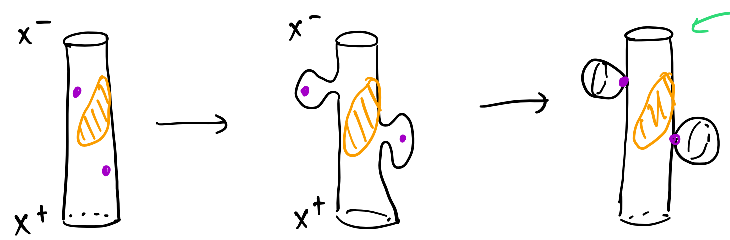
Prop (Bubbling) Let $u^\nu \in M(x^-, x^+; H, J)$ be a sequence s.t.

$$\sup_\nu E(u^\nu) < \infty .$$

Then $\exists$ finitely many pts $z_j = s_j + it_j \in \mathbb{R}\times S^1$, $j = 1, \dots, \ell$, and a solution u of Floer eq. s.t u^ν converges to u, uniformly with all derivatives on cpt subsets of $\mathbb{R}\times S^1 - \{z_1, \dots, z_\ell\}$.

Moreover

$$E(u) \leq \liminf_{\nu\to\infty} E(u^\nu) - \ell\hbar .$$



This limits might not be x^- and x^+.

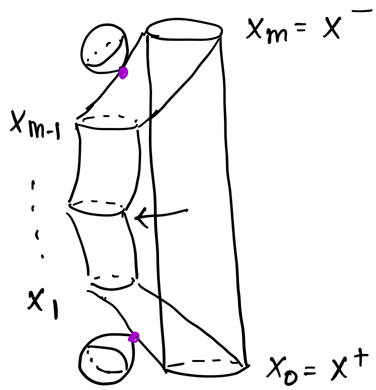
Cor: Suppose all periodic orbits in $P(H)$ are all nondeg.
Let $u^{\nu} \in \mathcal{M}(x^-, x^+; H, J)$ s.t. $\sup_{\nu} E(u^{\nu}) < \infty$.
Then there exists a subsequence and finitely many orbits $x_0, \dots, x_m \in P(H)$ with $x_0 = x^+$ and $x_m = x^-$ and finitely many connecting orbits

$$u_j \in \mathcal{M}(x_j, x_{j-1}; H, J) \qquad j = 1, \dots, m$$

and finitely many sequences s_j^{ν} s.t.
$u^{\nu}(s + s_j^{\nu}, t)$ converges modulo bubbling to $u_j(s,t)$.

Moreover, $\sum_{j=0}^{m} E(u_j) \leq \limsup_{\nu \to \infty} E(u^{\nu}) - l\hbar$

where l = total no. of bubbles.



* Floer homology

(M, ω) monotone, $H \in \mathcal{H}_{reg}$

Chain complex: $$CF_k(H) = \bigoplus_{\substack{x \in P(H) \\ \mu_{CZ}(x;H) = k \pmod{2N}}} \mathbb{Z}_2 \langle x \rangle.$$

Can choose $\mathbb{Z}$, $\mathbb{Q}$ instead of $\mathbb{Z}_2$.

Differential: $$\partial^F \langle y \rangle = \sum_{\substack{x \in P(H) \\ \mu_{CZ}(x;H) \\ = k-1 \pmod{2N}}} \sum_{[u] \in \hat{\mathcal{M}}^1(y, x; H, J)} \langle x \rangle.$$

$\mu_{CZ}(y;H) = k \pmod{2N}$

Thm (Floer) If (M,ω) is monotone and $H \in H_{reg}$

then $\partial^F \circ \partial^F = 0$.

- In the case that no bubbling happens we get that $\hat{M}^2(z,x) = M^2(z,x)/\mathbb{R}$ is a compact 1-manifold

with $\partial \hat{M}^2(z,x) = \bigcup_{y \in P(H)} \hat{M}^1(z,y) \times \hat{M}^1(y,x)$.

↪ need compactness + gluing

Hence, $\partial^2(z) = 0$.

- When is there no bubbling:

1. If $\omega(\pi_2(M)) = 0$.
2. If minimal chern no. $N \geq 2$.

$$\sum_{j=1}^{m} \mu(u_j) = \sum_{j=1}^{m} \left(\eta_H(x_j) - \eta_H(x_{j-1}) + 2\tau E(u_j)\right)$$

$$= \eta_H(x_-) - \eta_H(x_+) + 2\tau \sum_{j=1}^{m} E(u_j)$$

$$\leq \eta_H(x_-) - \eta_H(x_+) + 2\tau E(u^\nu) - 2\tau l \hbar \quad \text{(convergence modulo bubbling)}$$

$$= 2 - 2lN \quad (\tau\hbar = N).$$

$$\Rightarrow \quad m \leq \sum_{j=1}^{m} \mu(u_j) = 2 - 2lN \qquad N \geq 2$$

$$\leq 2 - 4l$$

$$\Rightarrow \quad m = 2$$

$l = 0$. $\therefore$ no bubbling.

3. If $N=1$, there may exist J-hol. spheres

$$c_1(v) = \int_{S^2} v^* c_1 = 1.$$

If such a sphere appears as a bubble, then the remaining limit solution u must have zero energy. Same index argument

$$\left[m \leq 2 - 2lN = 2 - 2l \ (\because N=1) \right.$$
$$\left. \Rightarrow \begin{array}{l} m=2 \\ l=0 \end{array} \text{ or } \begin{array}{l} m=0 \\ l=1 \end{array}. \right]$$

Hence, $u(s,t) = x(t) = z(t)$.

Fact 1: Bubbling sphere $v: S^2 \to M$ must intersect the remaining limit curve $u(s,t) = x(t)$.

Fact 2: For generic J, the J-hol. curves with Chern no. 1 form moduli space $\mathcal{M}(1;J) \subset \mathrm{Map}(S^2, M)$ of dimension $2n+2$.

- Conformal group $G = PSL(2,\mathbb{C}) \leftarrow$ 6 dim

  $\mathcal{M}(1;J)/G$ has dim $2n-4$

- Each sphere is 2 dim.

  $\mathcal{M}(1;J) \times S^2/G$ has dim $2n-2$

$\therefore$ Points on J-hol curves with Chern no. 1 are a codim 2 subset of M.

$= \text{image}\left(ev: \mathcal{M}(1;J) \times S^2 / G \to M\right)$

$\therefore$ For generic H, these spheres do not intersect 1-periodic orbits.

$\therefore$ no bubbles.

<u>Invariance</u>: for 2 pairs (H^α, J^α) and (H^β, J^β) satisfying regularity requirements

$\exists$ natural isom.

$$\Phi^{\beta\alpha} : HF_*(M, \omega, H^\alpha, J^\alpha) \to HF_*(M, \omega, H^\beta, J^\beta)$$

<u>Thm</u>: If $H_t = H$ and H is sufficiently C^2-small, then the only 1-periodic orbits of X_H are constant solutions and the 0- & 1-dimensional moduli spaces of Floer's connecting orbits consist entirely of gradient flow lines.

$$\therefore \quad HF_*(H, J) = HM_*(H)$$

<u>Cor</u>: $\quad \# P(H) \geq \sum_{k=0}^{2n} \beta_k$

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Prop: If $\|dX_H\|_{L^2} < 2\pi$, then the only solutions of period 1 of the Hamiltonian system ass. with H are the constant solutions

Pf: $x : S^1 \to M$ 1-periodic solution. This proof is for $M = \mathbb{R}^{2n}$.

$$x(t) = \sum_n c_n(x) e^{2\pi i n t}$$

$$\dot{x}(t) = \sum_n 2 n i \pi c_n(x) e^{2\pi i n t}$$

$$\ddot{x}(t) = \sum_n (-4\pi^2 n^2) c_n(x) e^{2\pi i n t}$$

Parseval's identity implies

$$\|\ddot{x}\|^2_{L^2} = \sum 4\pi^2 n^2 |c_n(\dot{x})|^2$$

$$\geq 4\pi^2 \sum_{n \neq 0} |c_n(\dot{x})|^2 = 4\pi^2 \|\dot{x}\|^2_{L^2}$$

$\because c_0(\dot{x}) = 0$.

$$\therefore \|\dot{x}\|_{L^2} \leq \frac{1}{2\pi} \|\ddot{x}\|_{L^2}$$

x solution $\Rightarrow$ $\ddot{x} = (dX_H) \cdot \dot{x}$

$+ \|dX_H\|_{L^2} < 2\pi \Rightarrow \|\ddot{x}\|_{L^2} < 2\pi \|\dot{x}\|_{L^2}$ if $\dot{x} \neq 0$

$\therefore \dot{x} = 0$ and x is constant solution.

Prop.: (M, ω) symplectic mfld.

$H : M \to \mathbb{R}$ Hamiltonian.

H sufficiently C^2-small $\Rightarrow \|X_H\|_{L^2} < 2\pi$.

Thm: $\exists$ nondegenerate, suff. C^2-small Ham. H for which $CF_*(H,J) = CM_{*+n}(H,J)$.

Fact: If $H : M \to \mathbb{R}$ time independent

$x \in \mathrm{Crit}(H)$

$\underset{\text{Morse}}{\mathrm{Ind}_H(x)} = \underset{\text{Maslov, CZ.}}{\mu(x)} + n$

Thm: Let $H : M \to \mathbb{R}$ be Morse.
$\exists$ dense subset of $J_{reg}(H) \cap J(\omega)$ s.t. $(H, -JX_H)$ is Morse-Smale.

- We show that for large enough $k \in \mathbb{N}$, the count of Floer cylinders for the Floer cx is exactly the count of the gradient traj for the Morse cx.

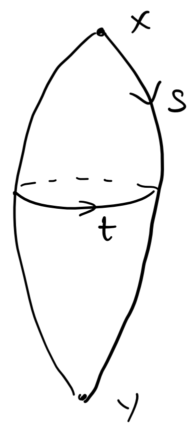
Suppose that the above is not true

Assume there exists $n_k \to \infty$

& solutions u_{n_k} (that depends on t)

that connects critical pts x and y

of Floer eqn.



$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \frac{1}{n_k}\nabla H = 0$$

Let $v_{n_k}(s,t) = u_{n_k}(n_k s, n_k t)$, is periodic of period $1/n_k$.

Then $v_{n_k} \in \mathcal{M}(x, y, H)$.

<u>Case</u>: $\mathrm{Ind}(x) - \mathrm{Ind}(y) = 1 = \mu(x) - \mu(y)$.

After extracting subsequence $v_{n_k} \to v \in \mathcal{M}(x, y, H)$.

If v does not depend on t,

then v lies in a component of $\mathcal{M}'(x, y, H)$

ie. v is an isolated point of $\widehat{\mathcal{M}}'(x, y, H)$

$\therefore$ For sufficiently large k

$v_{n_k}(s,t) = v(s+\sigma_k, t) = v(s+\sigma_k)$. $\Rightarrow\Leftarrow$

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TST v does not depend on t.

We use the fact that the period of v_{n_k} goes to 0.

Let $r \in \mathbb{R}$. Fix $s \in \mathbb{R}$ and $t \in [0,1]$.

v_{n_k} is periodic of period $\frac{1}{n_k}$,

$$v_{n_k}(s,t) = v_{n_k}\left(s, t + \frac{[r n_k]}{n_k}\right)$$ ← integral part

As $k \to \infty$, $\frac{[r n_k]}{n_k} \to r$

$\therefore$ $v_{n_k}(s,t) \to v(s,t)$

RHS $\longrightarrow v(s,t+r)$

$\therefore$ $v(s,t) = v(s,t+r)$ $\forall r$.

Case: $\mathrm{Ind}(x) - \mathrm{Ind}(y) = 2$ more involved

Talk 4 Lagrangian Floer Homology

(M,ω) symplectic manifold.

$(M\times\overline{M},\ \omega\oplus(-\omega))$ is a symplectic manifold.

$\Delta_M = \{(x,x) \mid x\in M\}$ is a Lagrangian.

let $\psi : M \to M$ be a symplectomorphism.

Then $\Gamma(\psi) = \{(x,\psi(x)) \mid x\in M\} \subset M\times\overline{M}$ is a Lagrangian.

$$\left\{\begin{matrix}\text{Fixed points of} \\ \psi\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Intersections of} \\ \Delta_M \cap \Gamma(\psi)\end{matrix}\right\} = I(\Delta_M,\psi)$$

Ref: Floer — Morse theory for Lagrangian Intersections

Thm (Floer) Let P be a cpt mfld and let $L\subset P$ be a compact Lagrangian submfld of P with $\pi_2(P,L)=0$.

Moreover let ϕ be a Hamiltonian diffeomorphism so that $\phi^1(L)\cap L$ transversely. Then to each $x\in I(L,\phi)$ we can assign an integer $\mu(x)$ s.t. the following holds:

Define $\displaystyle \Pi_\phi(t) = \sum_{x\in I(L,\phi)} t^{\mu(x)}$

$$\Pi_L(t) = \sum_{k=0}^{\dim L} \dim_{\mathbb{Z}_2} H^k(L,\mathbb{Z}_2)\, t^k .$$

Then $\exists\ Q\in\mathbb{Z}[t]$ with all positive coeff. s.t.

$$\Pi_\phi(T) = \Pi_L(t) + Q(t)(1+t).$$

Cor: Put $t=1$, $\displaystyle |L\cap\phi(L)| \geq \sum_{k=0}^{\dim L} \dim_{\mathbb{Z}_2} H^k(L;\mathbb{Z}_2)$.

<u>Note</u>: $\mathbb{Z}_2$- coefficients is not neccessary in the orientable case.

<u>Note</u>: If $L \cap \varphi_1(L)$ is not orientable, then one can estimate $|L \cap \varphi_1(L)|$ using "cuplength".

<u>Thm 2</u>: P cpt symplectic manifold with $\pi_2(P) = 0$ and φ Hamiltonian diffeo whose all fixed pts are non-degenerate, then

$$\# \operatorname{Fix}(\varphi) \geq \sum_{k=0}^{2n} \dim_{\mathbb{Z}_2} H_k(P; \mathbb{Z}_2)$$

Let
$$CF^p = \bigoplus_{\substack{x \in I(L,\varphi) \\ \mu(x) = p}} \mathbb{Z}_2 \langle x \rangle$$

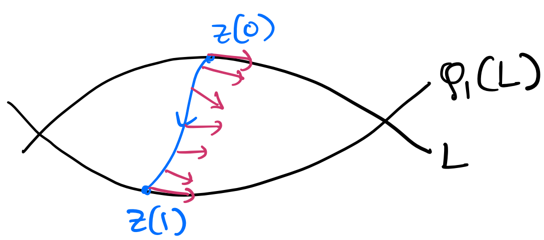
Thm 1 holds

$\Leftrightarrow$ $\exists$ a $\mathbb{Z}_2$-module homomorphism $\delta : CF^p \to CF^{p+1}$ s.t. $\delta\delta = 0$ and

$$\frac{\ker \delta}{\operatorname{im} \delta} = H^*(L; \mathbb{Z}_2).$$

Define

$$\Omega := \Omega(L, \phi) = \left\{ z \in C^\infty([0,1], P) \;\middle|\; \begin{array}{l} z(0) \in L \\ z(1) \in \phi_1(L) \\ [\varphi_t(z(t))] = 0 \in \pi_1(P, L) \end{array} \right\}$$



③

- Tangent space $T_z\Omega$ consists of tangent fields ξ of P along z that are tangent to L at 0 and $L' := \varphi_1(L)$ at 1.

- ω induces a "1-form"

$$\alpha(\xi) = \int_0^1 \omega(\dot{z}(t), \xi(t))\, dt \qquad \text{on } \Omega.$$

- $J \in \mathrm{End}(P)$ almost cx str. on P, ie. $J^2 = -\mathrm{Id}$.
s.t. $J \in \mathcal{J}(\omega)$ ie. $g = \omega(\cdot, J\cdot)$ is a metric on P.

- L^2-gradient of a wrt g is
$\mathrm{grad}_z = J\dot{z}$

$$\langle J\dot{z}, \xi\rangle := \int_0^1 g(J\dot{z}, \xi) = \int_0^1 \omega(\dot{z}, \xi) = da(z)\cdot\xi \;.$$

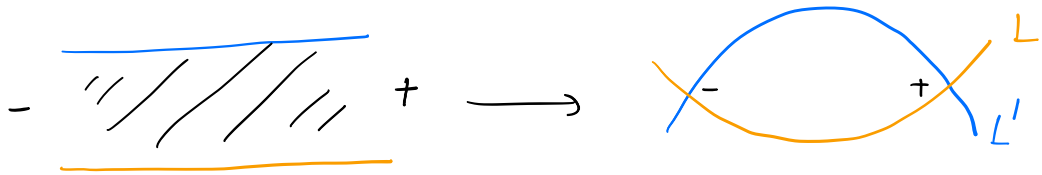
- Trajectory: $u : \mathbb{R} \to \Omega \qquad u(s)(t) = u(s,t)$

$$\bar{\partial} u := \frac{\partial u}{\partial \tau} + J\frac{\partial u}{\partial t} = 0$$

$\therefore$ trajectories are holomorphic maps

$$u : \Theta \to M$$
$$\Theta = \mathbb{R}\times[0,1] \cong \{z \in \mathbb{C} \mid 0 \le \mathrm{Im}\, z \le 1\}$$

$u(\{0\}\times\mathbb{R}) \subset L$
$u(\{1\}\times\mathbb{R}) \subset L'$



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- Index formula, $\exists$ a relative maslov ind
  $\mu : I(L, L') \to \mathbb{Z}$
  s.t. $\quad \operatorname{ind}(D_u) = \mu(x) - \mu(y)$.

- $y \in I(L, \phi)$

$$\delta(y) = \sum_{\mu(x) = \mu(y)+1} \#_{\mathbb{Z}_2} \left(\mathcal{M}(x,y)/\mathbb{R} \right) \langle x \rangle$$

$\mathbb{R} \curvearrowright \mathcal{M}(x,y)$ by translating s.

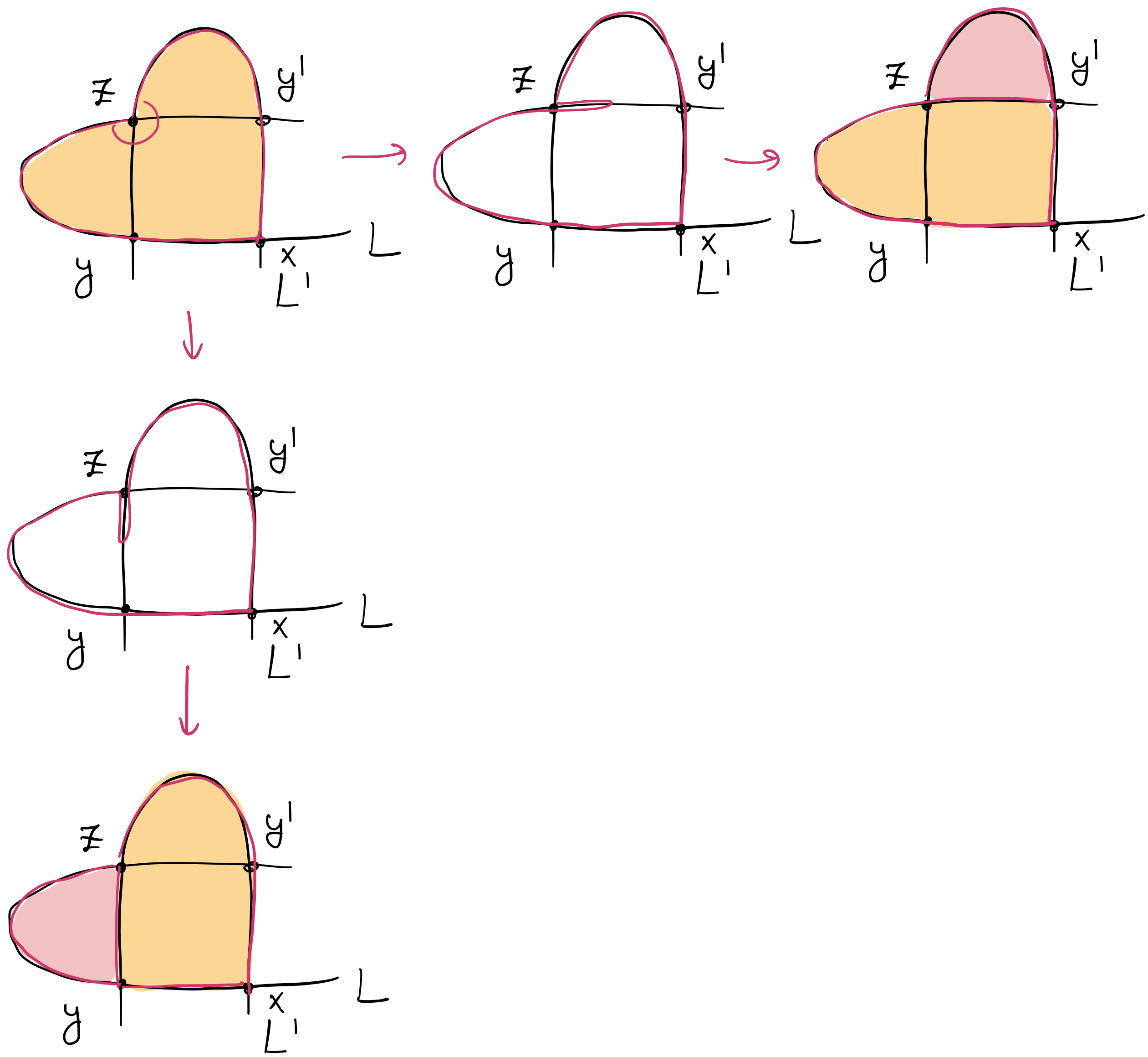
- Prop: $\quad \delta_p \delta_{p-1} = 0$.

$$\therefore \; HF^p(L, \phi; J) = \frac{\ker \delta_p}{\operatorname{Im} \delta_{p-1}}$$

<u>Thm</u>: • HF^p are independent of J and invariant upto deformation of L on $\varphi(L)$ by Hamiltonian diffeos.

• $HF^p(L, \varphi; J) \cong H^*(L; \mathbb{Z}_2)$.

* $\delta_p \delta_{p-1} = 0$



- If $\exists$ a Hamiltonian $H: M \to \mathbb{R}$ such that $\varphi_H^1(L) \cap L = \phi$, L is called displaceable.

We have show that if L is a monotone Lagrangian, and $HF(L) \neq 0$, L is not displaceable.

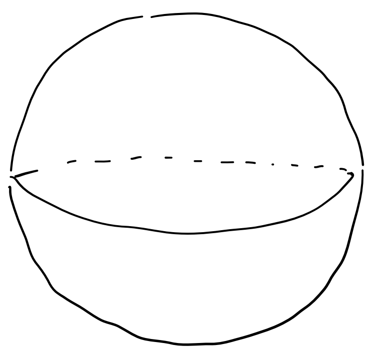
- Monotone Lagrangian: A Lagrangian $L \subseteq (M^{2n}, \omega)$ is monotone if $[\omega] = \lambda [\mu]$ for some $\lambda > 0$,

thinking of $[\omega], [\mu] : \pi_2(M, L) \to \mathbb{R}$.

μ = Maslov index = depends on the bundle

$$\begin{array}{c} TL \subset TM \\ \downarrow \\ L \end{array}$$

- Example: $(S^2 = \mathbb{CP}^1, \omega_0)$ $\qquad S^1_{eq} \subset S^2$



S^1_{eq} is monotone:

$\pi^2(S^2, S^1_{eq})$ is generated by the 2 disks. They have the same area

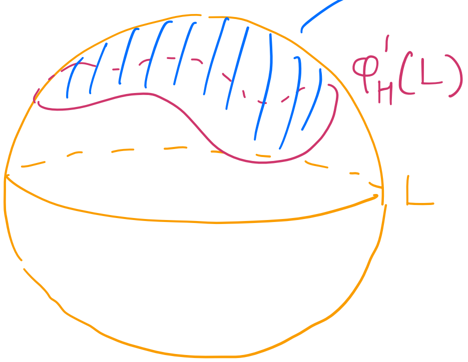
Any S^1 dividing S^2 into 2 disks of equal area is monotone.

- If $L \subset M$ is monotone, then $\delta^2 = 0$ on $CF(L,L)$ and $HF(L,L) \cong H_*(L)$.

Ex: S^1_{eq} is not displaceable.

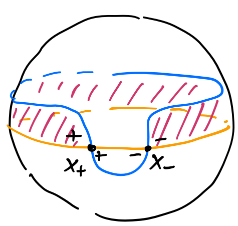
Turns out if any S^1 is contained in one-hemisphere, it is displaceable by a rotation.

Does not really need HF. Note a Hamiltonian flow preserves areas. So if φ^1_H displaces S^1_{eq}, then $\varphi^1_H(S^1_{eq})$ is contained completely in the north or south hemisphere.
φ^1_H takes the hemispheres to the 2 disks, which is not possible



area is strictly less than area of hemisphere.

* Can explicitly compute



$|x_-| = 1$
$|x_+| = 0$
$\partial x_- = 2x_+ = 0$
$\partial x_+ = 0$

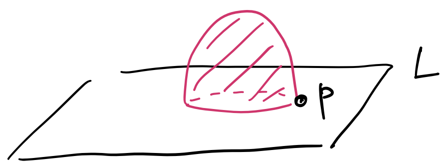
* Note that the definition of Lagrangian Floer th. did not use much about the fact that $L' = \varphi^1_H(L)$.

What happens if we consider L, L' two Lagrangians in M?

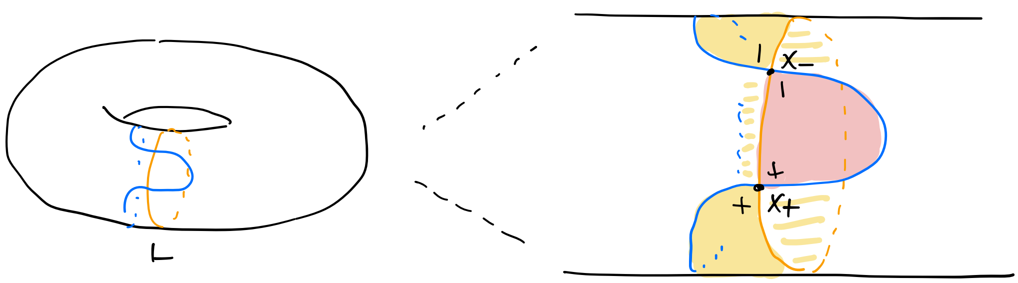
We can assume $\pi_2(M,L) = \pi_2(M,L') = 0$

$$\text{or } \omega|_{\pi_2(M,L)} \equiv \omega|_{\pi_2(M,L')} \equiv 0$$

(or L and L' are monotone and a count of holomorphic discs in M with boundary on L passing through a point $m_0(L) = m_0(L')$.)



Then $HF(L, L'; \mathbb{Z}_2)$ is unobstructed, ie. we can compute it.



$|x_-| = 1 \qquad |x_+| = 0 \qquad \partial x_- = 2x_+ = 0$

$\partial x_+ = 0$

$$HF_1(L, L) = \mathbb{Z}_2\langle x_- \rangle \cong \mathbb{Z}_2$$

$$HF_0(L, L) = \mathbb{Z}_2\langle x_+ \rangle \cong \mathbb{Z}_2$$

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